A quantitative extension of Interval Temporal Logic over infinite words

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Interval Temporal logics (ITLs)

- Alternative framework for reasoning about time with respect to popular Point-based Temporal Logics (PTLS) such as LTL, CTL, and CTL*.
- ITLs assume intervals, instead of points, as primitive temporal entities.
- Specification of relevant temporal properties (e.g. actions with duration and temporal aggregations) which cannot be naturally expressed by PTLs.
- Many application fields: e.g. reasoning about action and change, planning, specification and verification of programs, temporal and spatio-temporal databases.
- The landmark is Halpern and Shoham's modal logic of time intervals (HS).

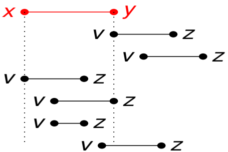
The logic HS

HS formulas φ over *AP*:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X \rangle \varphi$$

- $p \in AP$, $\langle X \rangle$ is the existential temporal modality for the (non-trivial) Allen's relation $\mathcal{R}_X$ with $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$ between pairs of intervals.
- The table illustrates six Allen's relations $\mathcal{R}_X$ with $X \in \{A, L, B, E, D, O\}$. $\mathcal{R}_{\overline{X}}$ is the inverse of $\mathcal{R}_X$.

Allen relation	HS	Definition w.r.t. interval structures	Example
MEETS	$\langle A \rangle$	$[x,y]\mathcal{R}_A[v,z] \iff y = v$	
BEFORE	$\langle L \rangle$	$[x,y]\mathcal{R}_L[v,z] \iff y < v$	
STARTED-BY	$\langle B \rangle$	$[x,y]\mathcal{R}_B[v,z] \iff x = v \wedge z < y$	
FINISHED-BY	$\langle E \rangle$	$[x,y]\mathcal{R}_E[v,z] \iff y = z \wedge x < v$	
CONTAINS	$\langle D \rangle$	$[x,y]\mathcal{R}_D[v,z] \iff x < v \wedge z < y$	
OVERLAPS	$\langle O \rangle$	$[x,y]\mathcal{R}_O[v,z] \iff x < v < y < z$	



Semantics of HS

Interpreted over interval structures $S = (LO, V)$: linear order LO + valuation V

- V assigns to each interval I over LO the propositions in AP holding at I.

$$S, I \models \langle X \rangle \varphi \Leftrightarrow \text{ for some interval } J \text{ with } \mathcal{R}_X(I, J),\ S, J \models \varphi$$

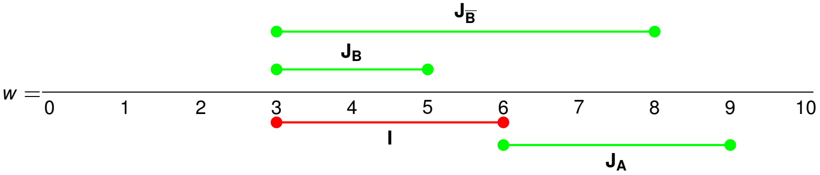
- Non-strict semantics with singleton intervals: the fragment $\mathrm{B\overline{B}E\overline{E}}$ is expressively complete.
- Over discrete linear orders such as $(N, <)$, HS is highly undecidable even for small fragments (with some meaningful exceptions).

HS under the infinite word semantics

Interpreted over homogeneous N-interval structures: the linear order is $(N, <)$ and

- Homogeneity assumption: a proposition $p \in AP$ holds over an interval I iff it holds over each subinterval of I.

Homogeneus N-interval structures correspond to infinite words over 2^{AP}.



$$
\begin{aligned}
I &\models_w p && \Leftrightarrow p \in w(i) \text{ for all } i \in I \\
J_A &\models_w \varphi && \Rightarrow I \models \langle A \rangle \varphi \\
J_B &\models_w \varphi && \Rightarrow I \models \langle B \rangle \varphi \\
J_{\overline{B}} &\models_w \varphi && \Rightarrow I \models \langle \overline{B} \rangle \varphi
\end{aligned}
$$

HS under the infinite word semantics

Known results in this setting:

L. Bozzelli et al. - Interval vs. Point Temporal Logic Model Checking: An Expressiveness Comparison - ACM Trans. Comput. Logic 2019

- HS and standard LTL expressively equivalent.
- Fragment AB expressively complete and exponentially more succinct than LTL.
  - Satisfiability and model checking of AB are **EXPSPACE**-complete.
- Open question: exact complexity of satisfiability and model checking for full logic.
  - Unique known upper bounds are non-elementary
    (even for the fragment BE of prefixes and suffixes over finite words).

Quantitative extensions of temporal logics

- Traditional PTLs only express *qualitative* requirements on ordering of events. A "responsiveness" requirement: every request p is followed by a response q"

$$G(p \longrightarrow Fq)$$

How soon the response should follow the request?

- Quantitative extensions of PTLs (e.g. MTL) equip temporal modalities with timing constraints for specifying integer bounds on the delays among events.

$$G(p \longrightarrow F_{\leq c}q)$$

- Modalities in MTL express integer bounds on the duration (length) of the interval with endpoints the current position and the position selected by the modality.

Quantitative extensions of HS have not been investigated in the literature.

Paper contribution

- A quantitative extension of HS, called *Difference HS*,under the infinite word semantics.
  - Generalizes MTL approach for discrete time.
  - Constrained modalities allows to quantitatively compare the durations of the current interval and the interval selected by the modality.
- Decidability and complexity issues for satisfiability and model checking of the novel logic and its meaningful fragments.
- Comparison in terms of expressiveness and succinctness of decidable fragments of the novel logic with (fragments of) linear-time hybrid logic.

Difference HS (DHS)

DHS formulas φ over AP:

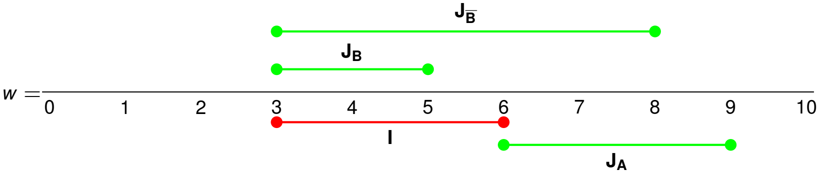
$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X \rangle \varphi \mid \langle X \rangle_{\Delta \sim c} \varphi$$

- $p \in AP$, $\sim \in \{<, \leq, =, >, \geq\}$, c is an integer encoded in binary, and
- $\langle X \rangle_{\Delta \sim c}$ is the existential constrained temporal modality for the Allen's relation $\mathcal{R}_X$ where $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$:
  - Δ denotes difference,
  - the constraint $\sim c$ refers to the difference between the lengths of two intervals, the one selected by the modality $\langle X \rangle$ and the current one.

DHS semantics over infinite words

Given an infinite word w over 2^{AP} and an interval $I \subseteq N$:

$$I \models_w \langle X \rangle_{\Delta \sim c} \varphi \Leftrightarrow \text{ for some interval } J \text{ such that } I \mathcal{R}_X J \text{ and } |J| - |I| \sim c, J \models_w \varphi$$



$$J_A \models_w \varphi \quad \Rightarrow I \models \langle A \rangle_{\Delta=0} \varphi$$

$$J_B \models_w \varphi \quad \Rightarrow I \models \langle B \rangle_{\Delta \leq -1} \varphi$$

$$J_{\overline{B}} \models_w \varphi \quad \Rightarrow I \models \langle \overline{B} \rangle_{\Delta \geq 1} \varphi$$

Meaningful fragments of DHS

Restriction to the set of allowed temporal modalities, and allowing or not equality constraints.

- DHS_{simple}: NO constrained modalities for $\mathcal{R}_A$, $\mathcal{R}_L$, $\mathcal{R}_O$, and their inverses.
- For Allen's relation $\mathcal{R}_X$, fragment DHS_X with constrained modalities for $\mathcal{R}_X$ only.
- For any subset of Allen's relations $\{\mathcal{R}_{X_1}, .., \mathcal{R}_{X_n}\}$, the fragment $\text{D}_{simple}(\text{X}_1 \ldots \text{X}_\text{n})$ of DHS_{simple} with temporal modalities for $\mathcal{R}_{X_1}, .., \mathcal{R}_{X_n}$ only.
- Monotonic versions of the previous fragments: no equality constraints.
  - $\langle \text{X} \rangle_{\Delta \prec c}$ with $\prec \in \{<, \leq\}$ is upward-monotone:

    $$\text{for } c' \geq c,\ I \models_w \langle \text{X} \rangle_{\Delta \prec c} \varphi \Rightarrow I \models_w \langle \text{X} \rangle_{\Delta \prec c'} \varphi$$

  - $\langle \text{X} \rangle_{\Delta \succ c}$ where $\succ \in \{>, \geq\}$ is downward-monotone:

    $$c' \leq c,\ I \models_w \langle \text{X} \rangle_{\Delta \succ c} \varphi \Rightarrow I \models_w \langle \text{X} \rangle_{\Delta \succ c'} \varphi$$

Undecidable fragments (over infinite words)

Theorem

Model checking and satisfiability for the fragment DHS_X of DHS, where $X \in \{A, L, O, \overline{A}, \overline{L}, \overline{O}\}$, are Σ^1_1-hard.

Undecidable fragments (over infinite words)

Theorem

Model checking and satisfiability for the fragment DHS_X of DHS, where $X \in \{A, L, O, \overline{A}, \overline{L}, \overline{O}\}$, are Σ^1_1-hard.

Proof.

By a reduction from the recurrent problem of Minsky two-counter machines M.

- The constrained versions of modalities $\langle \mathrm{A} \rangle$ (meets) and $\langle \mathrm{O} \rangle$ (overlap) and their inverses $\langle \overline{\mathrm{A}} \rangle$ and $\langle \overline{\mathrm{O}} \rangle$ can enforce that two adjacent intervals have the same length: natural encoding of recurrent M-computations.
- For the quantitative versions of modalities $\langle \mathrm{L} \rangle$ (before) and $\langle \overline{\mathrm{L}} \rangle$ (after), whose semantics is not local, a different and non-trivial encoding of recurrent M-computations is required.

□

DHS$_{simple}$: the maximal decidable fragment

DHS$_{simple}$: allows only constrained versions for modalities $\langle B \rangle$, $\langle D \rangle$, $\langle E \rangle$, and their inverses (the associated Allen's relations subsume the subset relation or its inverse).

- The maximal fragment not covered by the previous undecidability results.
- We will show that DHS$_{simple}$ is decidable and not more expressive than HS (over infinite words).
- In HS all modalities are expressible in terms of $\langle B \rangle$ (prefix), $\langle E \rangle$ (suffix), and their inverses $\langle \overline{B} \rangle$ and $\langle \overline{E} \rangle$. This does not hold in the quantitative setting!

Examples of DHS_{simple} specifications (over infinite words)

- Unlike HS, in DHS_{simple}, the requirement that a formula φ holds in the maximal proper sub-intervals of the current one can be succinctly expressed as:

$$(\langle E \rangle_{\Delta \geq -1}\, \varphi) \wedge (\langle B \rangle_{\Delta \geq -1}\, \varphi)$$

- Succinct encoding of constraints over the duration (length) of the current interval. Let $n > 0$:

$$\underbrace{\langle B \rangle_{\Delta \leq -n+1}\, \top}_{\text{Intervals of length at least } n}$$

$$\underbrace{\neg \langle B \rangle_{\Delta \leq -n}\, \top}_{\text{Intervals of length at most } n}$$

Decision procedures for DHS_{simple} and monotonic $\text{D}_{simple}(\text{AB}\overline{\text{B}})$

Intermediate step: linear-time translation of the given formulas into equivalent formulas of a novel extension of standard linear-time hybrid logic (HL).

- HL: LTL modalities + first-order constructs.
- HL and LTL have the same expressiveness.

Swap Constrained HL (SCHL)

SCHL formulas φ over *AP* and a set *X* of (position) variables:

$$\varphi ::= \ \mid p \mid x \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{F}\varphi \mid \mathsf{P}\varphi \mid \mathsf{F}_{\sim c}\varphi \mid \mathsf{P}_{\sim c}\varphi \mid {\downarrow}x.\varphi \mid \mathsf{swap}_x.\varphi$$

- $p \in AP$, $x \in X$, $\sim \in \{<, \leq =, \geq >\}$, and c is an integer (encoded in unary).
- $\mathsf{F}_{\sim c}$ and $\mathsf{P}_{\sim c}$: constrained versions of *strict eventually* and *past strict eventually*.
- ${\downarrow}x$ (downarrow modality): assigns variable x to current position.
- Novel modality swap_x: generalizes ${\downarrow}x$, swaps the x-value with the current position.

Semantics of Swap Constrained HL (SCHL)

Let w be an infinite word over 2^{AP}.
For a *valuation* g mapping assigning to each variable a position $i \geq 0$ the semantics is:

$$
\begin{array}{ll}
(w,i,g) \models x & \Leftrightarrow i = g(x) \\
(w,i,g) \models \mathsf{F}_{\sim c}\varphi & \Leftrightarrow \text{there is } j > i \text{ such that } j - i \sim c \text{ and } (w,j,g) \models \varphi \\
(w,i,g) \models \mathsf{P}_{\sim c}\varphi & \Leftrightarrow \text{there is } j < i \text{ such that } i - j \sim c \text{ and } (w,j,g) \models \varphi \\
(w,i,g) \models \downarrow x.\varphi & \Leftrightarrow (w,i,g[x \mapsto i]) \models \varphi \\
(w,i,g) \models \mathsf{swap}_x.\varphi & \Leftrightarrow (w,g(x),g[x \mapsto i]) \models \varphi
\end{array}
$$

Constrained modalities can be removed with a singly exponential blowup.

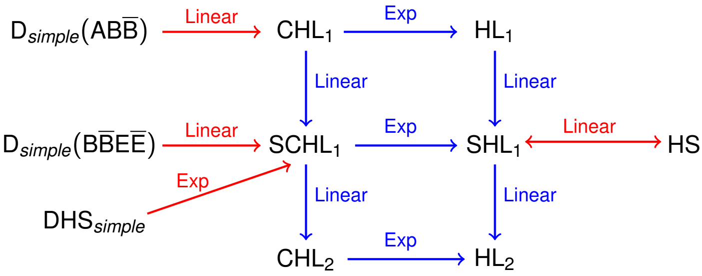
Fragments of SCHL and known results

- HL is SCHL without constrained and swap modalities.
- CHL is SCHL without swap modalities,
- SHL is SCHL without constrained modalities.
- The one-variable fragments HL_1, CHL_1, SHL_1, SCHL_1.
- The two-variable fragments HL_2 and CHL_2.
- Monotonic CHL_1 (CHL_1 without equality constraints).

Theorem (T. Schwentick et al., STACS'07)

- *Satisfiability of one-variable HL (HL_1) is* **EXPSPACE***-complete.*
- *Satisfiability of two-variable HL (HL_2) is already non-elementarily decidable.*

Translation between the considered logics



Remarks

- All logics have same expressiveness as LTL.
- SHL_1 lies between HL_1 (elementary) and HL_2 (non-elementary).
- SHL_1 is a characterization of HS over infinite words.
  - Open question: exact complexity of SHL_1 or HS satisfiability.
- Linear-time translation from monotonic $D_{simple}(AB\overline{B})$ into monotonic CHL_1.

Decision procedures for Monotonic CHL_1

Optimal automata-theoretic approach for satisfiability and model checking different from the one known for HL_1 (based on pebble automata):

- Exponential-time translation of monotonic CHL_1 formulas into equivalent Büchi two-way alternating automata on infinite words. Emptiness of these automata in known to be **PSPACE**-complete.
- Translation based on operational characterization of the satisfaction relation generalizing the standard tableaux construction for LTL.

Theorem

Model checking and satisfiability of monotonic CHL_1 are **EXPSPACE***-complete.*

Results for DHS_{simple} and monotonic $\mathrm{D}_{simple}(\mathrm{AB}\overline{\mathrm{B}})$

Monotonic $\mathrm{D}_{simple}(\mathrm{AB}\overline{\mathrm{B}})$ formulas can be translated in linear-time into equivalent monotonic CHL_1 formulas.

Theorem

- *Model checking and satisfiability of DHS_{simple} are decidable and* **2EXPSPACE**-*hard even for the fragment given by monotonic $\mathrm{D}_{simple}(\mathrm{ABE})$.*
- *Model checking and satisfiability of monotonic $\mathrm{D}_{simple}(\mathrm{AB}\overline{\mathrm{B}})$ are* **EXPSPACE**-*complete.*

Conclusions

- We have introduced a quantitative extension of HS under the infinite word semantics: modalities with constraints on the difference between the durations of the current interval and the one selected by the modality.
- Investigation of the decidability border of fragments using only a subset of the constrained modalities.
- **EXPSPACE**-completeness for a meaningful fragment capturing LTL.
- Characterization of HS over infinite words in terms of a fragment of linear-time hybrid logic (HL) which lies between the one-variable and the two-variable fragment of HL.

Thank you!